

# AdS/CFT and QCD

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The AdS/CFT correspondence between string theory in AdS space and conformal field theories in physical space-time leads to an analytic, semi-classical model for strongly-coupled QCD which has scale invariance and dimensional counting at short distances and color confinement at large distances. Although QCD is not conformally invariant, one can nevertheless use the mathematical representation of the conformal group in five-dimensional anti-de Sitter space to construct a first approximation to the theory. The AdS/CFT correspondence also provides insights into the inherently non-perturbative aspects of QCD, such as the orbital and radial spectra of hadrons and the form of hadronic wavefunctions. In particular, we show that there is an exact correspondence between the fifth-dimensional coordinate of AdS space  $z$  and a specific impact variable  $\zeta$  which measures the separation of the quark and gluonic constituents within the hadron in ordinary space-time. This connection allows one to compute the analytic form of the frame-independent light-front wavefunctions, the fundamental entities which encode hadron properties and allow the computation of decay constants, form factors, and other exclusive scattering amplitudes. New relativistic light-front equations in ordinary space-time are found which reproduce the results obtained using the 5-dimensional theory. The effective light-front equations possess remarkable algebraic structures and integrability properties. Since they are complete and orthonormal, the AdS/CFT model wavefunctions can also be used as a basis for the diagonalization of the full light-front QCD Hamiltonian, thus systematically improving the AdS/CFT approximation.

## 1. The Conformal Approximation to QCD

Quantum Chromodynamics, the Yang-Mills local gauge field theory of  $SU(3)_C$  color symmetry provides a fundamental understanding of hadron and nuclear physics in terms of quark and gluon degrees of freedom. However, because of its strong-coupling nature, it is difficult to find analytic solutions to QCD or to make precise predictions outside of its perturbative domain. An important goal is thus to find an initial approximation to QCD which is both analytically tractable and which can be systematically improved. For example, in quantum electrodynamics, the Schrödinger and Dirac equations provide accurate first approximations to atomic bound state problems which can then be systematically improved by using the Bethe-Salpeter formalism and correcting for quantum fluctuations, such as the Lamb Shift and vacuum polarization.

One of the most significant theoretical advances in recent years has been the application of the AdS/CFT correspondence [1] between string states defined on the 5-dimensional Anti-de Sitter (AdS) space-time and conformal field theories in physical space-time. The essential principle underlying the AdS/CFT approach to conformal gauge theories is the isomorphism of the group of Poincaré and conformal transformations  $SO(4, 2)$  to the group of isometries of Anti-de Sitter space. The AdS metric is

$$ds^2 = \frac{R^2}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2),$$

which is invariant under scale changes of the coordinate in the fifth dimension  $z \rightarrow \lambda z$  and  $x_\mu \rightarrow \lambda x_\mu$ . Thus one can match scale transformations of the theory in  $3 + 1$  physical space-time to scale transformations in the fifth dimension  $z$ .

QCD is not itself a conformal theory; however in the domain where the QCD coupling is approximately constant and quark masses can be neglected, QCD resembles a strongly-coupled conformal theory. As shown by Polchinski and Strassler [2], one can simulate confinement by imposing boundary conditions in the holographic variable at  $z = z_0 = 1/\Lambda_{QCD}$ . Confinement can also be introduced by modifying the AdS metric to mimic a confining potential. The resulting models, although *ad hoc*, provide a simple semi-classical approximation to QCD which has both counting-rule behavior [3–6] at short distances and confinement at large distances. This simple approach, which has been described as a “bottom-up” approach, has been successful in obtaining general properties of scattering amplitudes of hadronic bound states [2, 7–12] and the low-lying hadron spectra [13–20]. Studies of hadron couplings and chiral symmetry breaking [16, 21–24], quark potentials in confining backgrounds [25, 26] and pomeron physics [27, 28] has also been addressed within the bottom-up approach to holographic QCD, also known as AdS/QCD.

Recently, the behavior of the space-like form factors of the pion [29–31] and nucleons [32] has been discussed within the framework of AdS/QCD. Studies of geometric back-reaction controlling the infrared physics are given in

refs. [33, 34]. It is also remarkable that the dynamical properties of the quark-gluon plasma observed at RHIC [35] can be computed within the AdS/CFT correspondence [36]. In contrast to the simple bottom-up approach described above, the introduction of additional higher dimensional branes to the  $\text{AdS}_5 \times S^5$  background has been used to study chiral symmetry breaking [37], and recently baryonic properties by using D4-D8 brane constructs [38–40].

It was originally believed that the AdS/CFT mathematical tool would only be applicable to strictly conformal theories such as  $\mathcal{N} = 4$  supersymmetry. In our approach, we will apply AdS/CFT to the low momentum, strong coupling regime of QCD where the coupling is approximately constant. Theoretical [41] and phenomenological [42] evidence is in fact accumulating that the QCD couplings defined from physical observables such as  $\tau$  decay [43] become constant at small virtuality; *i.e.*, effective charges develop an infrared fixed point in contradiction to the usual assumption of singular growth in the infrared. Recent lattice gauge theory simulations [44] also indicate an infrared fixed point for QCD.

It is clear from a physical perspective that in a confining theory where gluons and quarks have an effective mass or maximal wavelength, all vacuum polarization corrections to the gluon self-energy must decouple at long wavelength; thus an infrared fixed point appears to be a natural consequence of confinement. Furthermore, if one considers a semi-classical approximation to QCD with massless quarks and without particle creation or absorption, then the resulting  $\beta$  function is zero, the coupling is constant, and the approximate theory is scale and conformal invariant.

In the case of hard exclusive reactions [6], the virtuality of the gluons exchanged in the underlying QCD process is typically much less than the momentum transfer scale  $Q$  since typically several gluons share the total momentum transfer. Since the coupling is probed in the conformal window, this kinematic feature can explain why the measured proton Dirac form factor scales as  $Q^4 F_1(Q^2) \simeq \text{const}$  up to  $Q^2 < 35 \text{ GeV}^2$  [45] with little sign of the logarithmic running of the QCD coupling.

One can also use conformal symmetry as a *template* [46], systematically correcting for its nonzero  $\beta$  function as well as higher-twist effects. For example, “commensurate scale relations” [47] which relate QCD observables to each other, such as the generalized Crewther relation [48], have no renormalization scale or scheme ambiguity and retain a convergent perturbative structure which reflects the underlying conformal symmetry of the classical theory. In general, the scale is set such that one has the correct analytic behavior at the heavy particle thresholds [49]. The importance of using an analytic effective charge [50] such as the pinch scheme [51, 52] for unifying the electroweak and strong couplings and forces is also important [53]. Thus conformal symmetry is a useful first approximant even for physical QCD.

In the AdS/CFT duality, the amplitude  $\Phi(z)$  represents the extension of the hadron into the compact fifth dimension. The behavior of  $\Phi(z) \rightarrow z^\Delta$  at  $z \rightarrow 0$  must match the twist-dimension of the hadron at short distances  $x^2 \rightarrow 0$ . As we shall discuss, one can use holography to map the amplitude  $\Phi(z)$  describing the hadronic state in the fifth dimension of Anti-de Sitter space  $\text{AdS}_5$  to the light-front wavefunctions  $\psi_{n/h}$  of hadrons in physical space-time [19], thus providing a relativistic description of hadrons in QCD at the amplitude level. In fact, there is an exact correspondence between the fifth-dimensional coordinate of anti-de Sitter space  $z$  and a specific impact variable  $\zeta$  in the light-front formalism which measures the separation of the constituents within the hadron in ordinary space-time. We derive this correspondence by noticing that the mapping of  $z \rightarrow \zeta$  analytically transforms the expression for the form factors in AdS/CFT to the exact QCD Drell-Yan-West expression in terms of light-front wavefunctions.

Light-front wavefunctions are relativistic and frame-independent generalizations of the familiar Schrödinger wavefunctions of atomic physics, but they are determined at fixed light-cone time  $\tau = t + z/c$ —the “front form” advocated by Dirac—rather than at fixed ordinary time  $t$ . An important advantage of light-front quantization is the fact that it provides exact formulas to write down matrix elements as a sum of bilinear forms, which can be mapped into their AdS/CFT counterparts in the semi-classical approximation. One can thus obtain not only an accurate description of the hadron spectrum for light quarks, but also a remarkably simple but realistic model of the valence wavefunctions of mesons, baryons, and glueballs. The light-front wavefunctions predicted by AdS/QCD have many phenomenological applications ranging from exclusive  $B$  and  $D$  decays, deeply virtual Compton scattering and exclusive reactions such as form factors, two-photon processes, and two-body scattering. One thus obtains a connection between the theories and tools used in string theory and the fundamental phenomenology of hadrons.

## 2. Light-Front Wavefunctions in Impact Space

The light-front expansion is constructed by quantizing QCD at fixed light-cone time [54]  $\tau = t + z/c$  and forming the invariant light-front Hamiltonian:  $H_{LF}^{QCD} = P^+ P^- - \vec{P}_\perp^2$  where  $P^\pm = P^0 \pm P^z$  [55]. The momentum generators  $P^+$  and  $\vec{P}_\perp$  are kinematical; *i.e.*, they are independent of the interactions. The generator  $P^- = i \frac{d}{d\tau}$  generates light-cone time translations, and the eigen-spectrum of the Lorentz scalar  $H_{LF}^{QCD}$  gives the mass spectrum of the color-singlet hadron states in QCD; the projection of the eigensolution on the free Fock basis gives the hadronic light-front wavefunctions.

The holographic mapping of hadronic LFWFs to AdS string modes is most transparent when one uses the impact parameter space representation [56]. The total position coordinate of a hadron or its transverse center of momentum  $\mathbf{R}_\perp$ , is defined in terms of the energy momentum tensor  $T^{\mu\nu}$

$$\mathbf{R}_\perp = \frac{1}{P^+} \int dx^- \int d^2 \mathbf{x}_\perp T^{++} \mathbf{x}_\perp. \quad (1)$$

In terms of partonic transverse coordinates

$$x_i \mathbf{r}_{\perp i} = x_i \mathbf{R}_\perp + \mathbf{b}_{\perp i}, \quad (2)$$

where the  $\mathbf{r}_{\perp i}$  are the physical transverse position coordinates and  $\mathbf{b}_{\perp i}$  frame independent internal coordinates, conjugate to the relative coordinates  $\mathbf{k}_{\perp i}$ . Thus,  $\sum_i \mathbf{b}_{\perp i} = 0$  and  $\mathbf{R}_\perp = \sum_i x_i \mathbf{r}_{\perp i}$ . The LFWF  $\psi_n(x_j, \mathbf{k}_{\perp j})$  can be expanded in terms of the  $n-1$  independent coordinates  $\mathbf{b}_{\perp j}$ ,  $j = 1, 2, \dots, n-1$

$$\psi_n(x_j, \mathbf{k}_{\perp j}) = (4\pi)^{\frac{(n-1)}{2}} \prod_{j=1}^{n-1} \int d^2 \mathbf{b}_{\perp j} \exp\left(i \sum_{j=1}^{n-1} \mathbf{b}_{\perp j} \cdot \mathbf{k}_{\perp j}\right) \tilde{\psi}_n(x_j, \mathbf{b}_{\perp j}). \quad (3)$$

The normalization is defined by

$$\sum_n \prod_{j=1}^{n-1} \int dx_j d^2 \mathbf{b}_{\perp j} \left| \tilde{\psi}_n(x_j, \mathbf{b}_{\perp j}) \right|^2 = 1. \quad (4)$$

One of the important advantages of the light-front formalism is that current matrix elements can be represented without approximation as overlaps of light-front wavefunctions. In the case of the elastic space-like form factors, the matrix element of the  $J^+$  current only couples Fock states with the same number of constituents. If the charged parton  $n$  is the active constituent struck by the current, and the quarks  $i = 1, 2, \dots, n-1$  are spectators, then the Drell-Yan West formula [57–59] in impact space is

$$F(q^2) = \sum_n \prod_{j=1}^{n-1} \int dx_j d^2 \mathbf{b}_{\perp j} \exp\left(i \mathbf{q}_\perp \cdot \sum_{j=1}^{n-1} x_j \mathbf{b}_{\perp j}\right) \left| \tilde{\psi}_n(x_j, \mathbf{b}_{\perp j}) \right|^2, \quad (5)$$

corresponding to a change of transverse momenta  $x_j \mathbf{q}_\perp$  for each of the  $n-1$  spectators. This is a convenient form for comparison with AdS results, since the form factor is expressed in terms of the product of light-front wave functions with identical variables.

We can now establish an explicit connection between the AdS/CFT and the LF formulae. It is useful to express (5) in terms of an effective single particle transverse distribution  $\tilde{\rho}$  [19]

$$F(q^2) = 2\pi \int_0^1 dx \frac{(1-x)}{x} \int \zeta d\zeta J_0\left(\zeta q \sqrt{\frac{1-x}{x}}\right) \tilde{\rho}(x, \zeta), \quad (6)$$

where we have introduced the variable

$$\zeta = \sqrt{\frac{x}{1-x}} \left| \sum_{j=1}^{n-1} x_j \mathbf{b}_{\perp j} \right|, \quad (7)$$

representing the  $x$ -weighted transverse impact coordinate of the spectator system. On the other hand, the expression for the form factor in AdS space is represented as the overlap in the fifth dimension coordinate  $z$  of the normalizable modes dual to the incoming and outgoing hadrons,  $\Phi_P$  and  $\Phi_{P'}$ , with the non-normalizable mode,  $J(Q, z) = zQK_1(zQ)$ , dual to the external source [8]

$$F(Q^2) = R^3 \int \frac{dz}{z^3} \Phi_{P'}(z) J(Q, z) \Phi_P(z). \quad (8)$$

If we compare (6) in impact space with the expression for the form factor in AdS space (8) for arbitrary values of  $Q$  using the identity

$$\int_0^1 dx J_0\left(\zeta Q \sqrt{\frac{1-x}{x}}\right) = \zeta Q K_1(\zeta Q), \quad (9)$$

then we can identify the spectator density function appearing in the light-front formalism with the corresponding AdS density

$$\tilde{\rho}(x, \zeta) = \frac{R^3}{2\pi} \frac{x}{1-x} \frac{|\Phi(\zeta)|^2}{\zeta^4}. \quad (10)$$

Equation (10) gives a precise relation between string modes  $\Phi(\zeta)$  in  $\text{AdS}_5$  and the QCD transverse charge density  $\tilde{\rho}(x, \zeta)$ . The variable  $\zeta$  represents a measure of the transverse separation between point-like constituents, and it is also the holographic variable  $z$  characterizing the string scale in AdS. Consequently the AdS string mode  $\Phi(z)$  can be regarded as the probability amplitude to find  $n$  partons at transverse impact separation  $\zeta = z$ . Furthermore, its eigenmodes determine the hadronic spectrum [19].

In the case of a two-parton constituent bound state, the correspondence between the string amplitude  $\Phi(z)$  and the light-front wave function  $\tilde{\psi}(x, \mathbf{b})$  is expressed in closed form[19]

$$|\tilde{\psi}(x, \zeta)|^2 = \frac{R^3}{2\pi} x(1-x) \frac{|\Phi(\zeta)|^2}{\zeta^4}, \quad (11)$$

where  $\zeta^2 = x(1-x)\mathbf{b}_\perp^2$ . Here  $b_\perp$  is the impact separation and Fourier conjugate to  $k_\perp$ .

### 3. Holographic Light-Front Representation

The equations of motion in AdS space can be recast in the form of a light-front Hamiltonian equation [55]

$$H_{LC} |\psi_h\rangle = \mathcal{M}^2 |\psi_h\rangle, \quad (12)$$

a remarkable result which allows the discussion of the AdS/CFT solutions in terms of light-front equations in physical 3+1 space-time. By substituting  $\phi(\zeta) = \left(\frac{\zeta}{R}\right)^{-3/2} \Phi(\zeta)$ , in the AdS wave equation describing the propagation of scalar modes in AdS space

$$[z^2 \partial_z^2 - (d-1)z \partial_z + z^2 \mathcal{M}^2 - (\mu R)^2] \Phi(z) = 0, \quad (13)$$

we find an effective Schrödinger equation as a function of the weighted impact variable  $\zeta$

$$\left[ -\frac{d^2}{d\zeta^2} + V(\zeta) \right] \phi(\zeta) = \mathcal{M}^2 \phi(\zeta), \quad (14)$$

with the effective potential  $V(\zeta) \rightarrow -(1-4L^2)/4\zeta^2$  in the conformal limit, where we identify  $\zeta$  with the fifth dimension  $z$  of AdS space:  $\zeta = z$ . We have written above  $(\mu R)^2 = -4 + L^2$ . The solution to (14) is  $\phi(z) = z^{-\frac{3}{2}} \Phi(z) = C z^{\frac{1}{2}} J_L(z\mathcal{M})$ . This equation reproduces the AdS/CFT solutions for mesons with relative orbital angular momentum  $L$ . The holographic hadronic light-front wave functions  $\phi(\zeta) = \langle \zeta | \psi_h \rangle$  are normalized according to

$$\langle \psi_h | \psi_h \rangle = \int d\zeta |\langle \zeta | \psi_h \rangle|^2 = 1, \quad (15)$$

and represent the probability amplitude to find  $n$ -partons at transverse impact separation  $\zeta = z$ . Its eigenmodes determine the hadronic mass spectrum.

The lowest stable state  $L = 0$  is determined by the Breitenlohner-Freedman bound[60]. Its eigenvalues are set by the boundary conditions at  $\phi(z = 1/\Lambda_{\text{QCD}}) = 0$  and are given in terms of the roots of Bessel functions:  $\mathcal{M}_{L,k} = \beta_{L,k} \Lambda_{\text{QCD}}$ . Normalized LFWFs  $\tilde{\psi}_{L,k}$  follow from (11)

$$\tilde{\psi}_{L,k}(x, \zeta) = B_{L,k} \sqrt{x(1-x)} J_L(\zeta \beta_{L,k} \Lambda_{\text{QCD}}) \theta(z \leq \Lambda_{\text{QCD}}^{-1}), \quad (16)$$

where  $B_{L,k} = \Lambda_{\text{QCD}} / \sqrt{\pi} J_{1+L}(\beta_{L,k})$ . The resulting wavefunctions depicted in Fig. 1 display confinement at large interquark separation and conformal symmetry at short distances, reproducing dimensional counting rules for hard exclusive processes and the scaling and conformal properties of the LFWFs at high relative momenta in agreement with perturbative QCD.

Since they are complete and orthonormal, these AdS/CFT model wavefunctions can be used as an initial ansatz for a variational treatment or as a basis for the diagonalization of the light-front QCD Hamiltonian. We are now in fact investigating this possibility with J. Vary and A. Harinandath. Alternatively one can introduce confinement by adding a harmonic oscillator potential  $\kappa^4 z^2$  to the conformal kernel in Eq. (14). One can also introduce nonzero quark masses for the meson. The procedure is straightforward in the  $\mathbf{k}_\perp$  representation by using the substitution

$$\frac{\mathbf{k}_\perp^2}{x(1-x)} \rightarrow \frac{\mathbf{k}_\perp^2 + m_1^2}{x} + \frac{\mathbf{k}_\perp^2 + m_2^2}{1-x}.$$

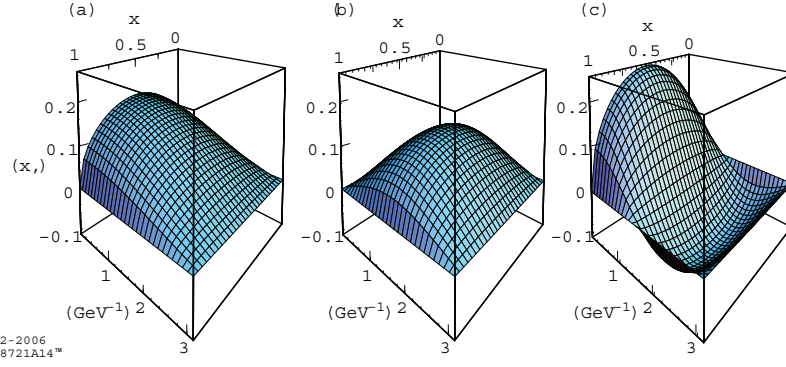


Figure 1: AdS/QCD Predictions for the light-front wavefunctions of a meson.

#### 4. Integrability of AdS/CFT Equations

The integrability methods of Ref. [[61]] find a remarkable application in the AdS/CFT correspondence. Integrability follows if the equations describing a physical model can be factorized in terms of linear operators. These ladder operators then generate all the eigenfunctions once the lowest mass eigenfunction is known. In holographic QCD, the conformally invariant  $3 + 1$  light-front differential equations can be expressed as ladder operators and their solutions can then be expressed in terms of analytical functions.

In the conformal limit the ladder algebra for bosonic ( $B$ ) or fermionic ( $F$ ) modes is given in terms of the operator ( $\Gamma^B = 1$ ,  $\Gamma^F = \gamma_5$ )

$$\Pi_\nu^{B,F}(\zeta) = -i \left( \frac{d}{d\zeta} - \frac{\nu + \frac{1}{2}}{\zeta} \Gamma^{B,F} \right), \quad (17)$$

and its adjoint

$$\Pi_\nu^{B,F}(\zeta)^\dagger = -i \left( \frac{d}{d\zeta} + \frac{\nu + \frac{1}{2}}{\zeta} \Gamma^{B,F} \right), \quad (18)$$

with commutation relations

$$[\Pi_\nu^{B,F}(\zeta), \Pi_\nu^{B,F}(\zeta)^\dagger] = \frac{2\nu + 1}{\zeta^2} \Gamma^{B,F}. \quad (19)$$

For  $\nu \geq 0$  the Hamiltonian is written as a bilinear form  $H_{LC}^{B,F} = \Pi_\nu^{B,F\dagger} \Pi_\nu^{B,F}$ . In the fermionic case the eigenmodes also satisfy a first order LF Dirac equation. For bosonic modes, the lowest stable state  $\nu = 0$  corresponds to the Breitenlohner-Freedman bound. Higher orbital states are constructed from the  $L$ -th application of the raising operator  $a^\dagger = -i\Pi^B$  on the ground state.

#### 5. Hadronic Spectra in AdS/QCD

The holographic model based on truncated AdS space can be used to obtain the hadronic spectrum of light quark  $q\bar{q}$ ,  $qqq$  and  $gg$  bound states. Specific hadrons are identified by the correspondence of the amplitude in the fifth dimension with the twist dimension of the interpolating operator for the hadron's valence Fock state, including its orbital angular momentum excitations. Bosonic modes with conformal dimension  $2 + L$  are dual to the interpolating operator  $\mathcal{O}_{\tau+L}$  with  $\tau = 2$ . For fermionic modes  $\tau = 3$ . For example, the set of three-quark baryons with spin  $1/2$  and higher is described by the light-front Dirac equation

$$(\alpha \Pi^F(\zeta) - \mathcal{M}) \psi(\zeta) = 0, \quad (20)$$

where  $i\alpha = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$  in the Weyl representation. The solution is

$$\psi(\zeta) = C \sqrt{\zeta} [J_{L+1}(\zeta \mathcal{M}) u_+ + J_{L+2}(\zeta \mathcal{M}) u_-], \quad (21)$$

with  $\gamma_5 u_{\pm} = u_{\pm}$ . A discrete four-dimensional spectrum follows when we impose the boundary condition  $\psi_{\pm}(\zeta = 1/\Lambda_{\text{QCD}}) = 0$ :  $\mathcal{M}_{\alpha,k}^+ = \beta_{\alpha,k}\Lambda_{\text{QCD}}$ ,  $\mathcal{M}_{\alpha,k}^- = \beta_{\alpha+1,k}\Lambda_{\text{QCD}}$ , with a scale-independent mass ratio[15].

Figure 2(a) shows the predicted orbital spectrum of the nucleon states and Fig. 2(b) the  $\Delta$  orbital resonances. The spin-3/2 trajectories are determined from the corresponding Rarita-Schwinger equation. The solution of the spin-3/2 for polarization along Minkowski coordinates,  $\psi_{\mu}$ , is similar to the spin-1/2 solution. The data for the baryon spectra are from [[64]]. The internal parity of states is determined from the SU(6) spin-flavor symmetry. Since only one parameter, the QCD mass scale  $\Lambda_{\text{QCD}}$ , is introduced, the agreement with the pattern of physical

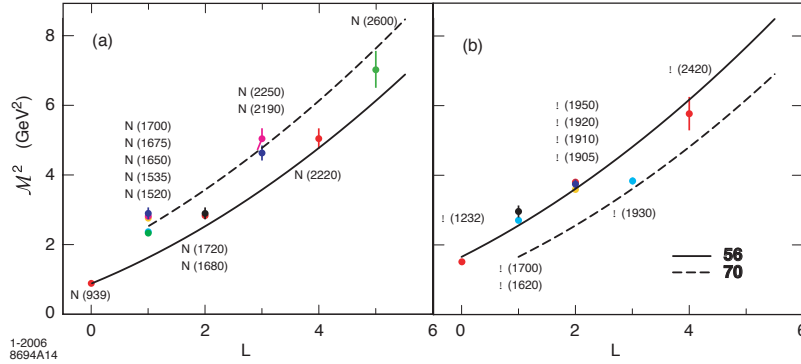


Figure 2: Predictions for the light baryon orbital spectrum for  $\Lambda_{\text{QCD}} = 0.25$  GeV. The **56** trajectory corresponds to  $L$  even  $P = +$  states, and the **70** to  $L$  odd  $P = -$  states.

states is remarkable. In particular, the ratio of  $\Delta$  to nucleon trajectories is determined by the ratio of zeros of Bessel functions. The predicted mass spectrum in the truncated space model is linear  $M \propto L$  at high orbital angular momentum, in contrast to the quadratic dependence  $M^2 \propto L$  in the usual Regge parameterization. One can obtain  $M^2 \propto (L+n)$  dependence in the holographic model by the introduction of a harmonic potential  $\kappa^2 z^2$  in the AdS wave equations [18]. This result can also be obtained by extending the conformal algebra written above. An account of the extended algebraic holographic model and a possible supersymmetric connection between the bosonic and fermionic operators used in the holographic construction will be described elsewhere.

## 6. Pion Form Factor

Hadron form factors can be predicted from the overlap integral representation in AdS space or equivalently by using the Drell-Yan West formula in physical space-time. For the pion string mode  $\Phi$  in the harmonic oscillator model [18]

$$\Phi_{\pi}^{HO}(z) = \frac{\sqrt{2}\kappa}{R^{3/2}} z^2 e^{-\kappa^2 z^2/2}, \quad (22)$$

the form factor has a closed form solution

$$F(Q^2) = 1 + \frac{Q^2}{4\kappa^2} \exp\left(-\frac{Q^2}{4\kappa^2}\right) Ei\left(-\frac{Q^2}{4\kappa^2}\right), \quad (23)$$

where  $Ei$  is the exponential integral

$$Ei(-x) = \int_{\infty}^x e^{-t} \frac{dt}{t}. \quad (24)$$

Expanding the function  $Ei(-x)$  for large arguments, we find for  $-Q^2 \gg \kappa^2$

$$F(Q^2) \rightarrow \frac{4\kappa^2}{Q^2}, \quad (25)$$

and we recover the dimensional counting rule. The prediction for the pion form factor is shown in Fig. 3. The space-like behavior of the pion form factor in the harmonic oscillator (HO) model is almost indistinguishable from the truncated-space (TS) model result. The form factor at high  $Q^2$  receives contributions from small  $\zeta$ , corresponding

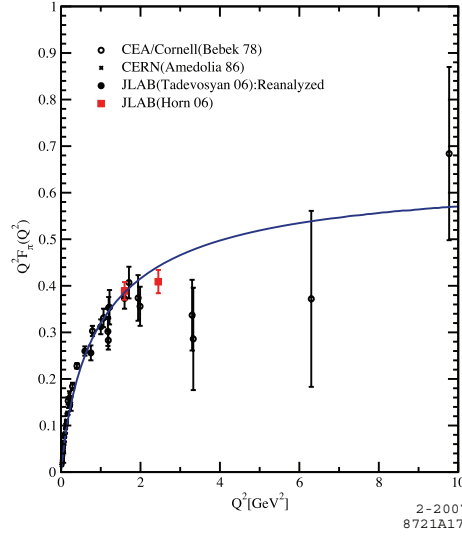


Figure 3:  $Q^2 F_\pi(Q^2)$  in the harmonic oscillator model for  $\kappa = 0.4$  GeV.

to small  $\vec{b}_\perp \sim \mathcal{O}(1/Q)$  (high relative  $\vec{k}_\perp \sim \mathcal{O}(Q)$ ), as well as  $x \rightarrow 1$ . The AdS/CFT dynamics is thus distinct from endpoint models [29] in which the LFWF is evaluated solely at small transverse momentum or large impact separation.

The  $x \rightarrow 1$  endpoint domain is often referred to as a “soft” Feynman contribution. In fact  $x \rightarrow 1$  for the struck quark requires that all of the spectators have  $x = k^+/P^+ = (k^0 + k^z)/P^+ \rightarrow 0$ ; this in turn requires high longitudinal momenta  $k^z \rightarrow -\infty$  for all spectators – unless one has both massless spectator quarks  $m \equiv 0$  with zero transverse momentum  $k_\perp \equiv 0$ , which is a regime of measure zero. If one uses a covariant formalism, such as the Bethe-Salpeter theory, then the virtuality of the struck quark becomes infinitely spacelike:  $k_F^2 \sim -\frac{k_\perp^2 + m^2}{1-x}$  in the endpoint domain. Thus, actually,  $x \rightarrow 1$  corresponds to high relative longitudinal momentum; it is as hard a domain in the hadron wavefunction as high transverse momentum.

## 7. The Pion Decay Constant

The pion decay constant is given by the matrix element of the axial isospin current  $J^{\mu 5a}$  between a physical pion and the vacuum state [62]  $\langle 0 | J_W^+(0) | \pi^-(P^+, \vec{P}_\perp) \rangle$ , where  $J_W^+$  is the flavor changing weak current. Only the valence state with  $L_z = 0$ ,  $S_z = 0$ , contributes to the decay of the  $\pi^\pm$ . Expanding the hadronic initial state in the decay amplitude into its Fock components we find

$$f_\pi = 2\sqrt{N_C} \int_0^1 dx \int \frac{d^2 \vec{k}_\perp}{16\pi^3} \psi_{\bar{q}q/\pi}(x, k_\perp). \quad (26)$$

This light-cone equation allows the exact computation of the pion decay constant in terms of the valence pion light-front wave function [6].

The meson distribution amplitude  $\phi(x, Q)$  is defined as [63]

$$\phi(x, Q) = \int^{Q^2} \frac{d^2 \vec{k}_\perp}{16\pi^3} \psi(x, k_\perp). \quad (27)$$

It follows that

$$\phi_\pi(x, Q \rightarrow \infty) = \frac{4}{\sqrt{3}\pi} f_\pi \sqrt{x(1-x)}, \quad (28)$$

with

$$f_\pi = \frac{1}{8} \sqrt{\frac{3}{2}} R^{3/2} \lim_{\zeta \rightarrow 0} \frac{\Phi(\zeta)}{\zeta^2}, \quad (29)$$

since  $\phi(x, Q \rightarrow \infty) \rightarrow \tilde{\psi}(x, \vec{b}_\perp \rightarrow 0)/\sqrt{4\pi}$  and  $\Phi_\pi \sim \zeta^2$  as  $\zeta \rightarrow 0$ . The pion decay constant depends only on the behavior of the AdS string mode near the asymptotic boundary,  $\zeta = z = 0$  and the mode normalization. For the truncated-space (TS) pion mode we find  $f_\pi^{TS} = \frac{\sqrt{3}}{8J_1(\beta_{0,k})} \Lambda_{\text{QCD}} = 83.4 \text{ MeV}$ , for  $\Lambda_{\text{QCD}} = 0.2 \text{ MeV}$ . The corresponding result for the transverse harmonic oscillator (HO) pion mode (22) is  $f_\pi^{HO} = \frac{\sqrt{3}}{8} \kappa = 86.6 \text{ MeV}$ , for  $\kappa = 0.4 \text{ GeV}$ . The values of  $\Lambda_{\text{QCD}}$  and  $\kappa$  are determined from the space-like form factor data as discussed above. The experimental result for  $f_\pi$  is extracted from the rate of weak  $\pi$  decay and has the value  $f_\pi = 92.4 \text{ MeV}$  [64].

It is interesting to note that the pion distribution amplitude predicted by AdS/QCD (28) has a quite different  $x$ -behavior than the asymptotic distribution amplitude predicted from the PQCD evolution [63] of the pion distribution amplitude  $\phi_\pi(x, Q \rightarrow \infty) = \sqrt{3}f_\pi x(1-x)$ . The broader shape of the pion distribution increases the magnitude of the leading twist perturbative QCD prediction for the pion form factor by a factor of 16/9 compared to the prediction based on the asymptotic form, bringing the PQCD prediction close to the empirical pion form factor [31].

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